JOURNAL OF APPROXIMATION THEORY 15, 11–17 (1975)

On Approximation by Semigroups of Nonlinear Contractions. II

U. WESTPHAL

Lehrstuhl A für Mathematik, Rheinisch-Westfälische Technische Hochschule Aachen, 51 Aachen, West Germany

Communicated by P. L. Butzer

This note is a sequel to the preceding one with the same title published in this *Journal* (13, 205–217 (1975)). The contents of the first paper are assumed to be known. References are in alphabetical order in each paper, but they, as well as the sections, are numbered consecutively.

6. A Particular Nonlinear Semigroup Associated with a Cauchy Problem

As a first application of the theory presented in part I, we treat an example considered by Y. Konishi [18]. Motivated by a problem related to the burning of gas in a rocket (see [27]), Konishi discussed the following initial value problem (for a similar problem, see [32]):

$$\begin{gathered} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - F\left(\frac{\partial u(x,t)}{\partial x}\right) \qquad (-\pi \leqslant x \leqslant \pi, t > 0), \\ u(-\pi,t) = u(\pi,t), \qquad \frac{\partial u(-\pi,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} \qquad (t > 0), \\ u(x,0) = f(x) \qquad (-\pi \leqslant x \leqslant \pi), \end{gathered} \tag{6.1}$$

where $F(v)$ is a continuous function on $\mathbb{R}^1$ such that $F(0) = 0$. He proved the existence of a unique solution of problem (6.1) by studying the nonlinear semigroup associated with it. In this note we wish to characterize the approximation behavior of this semigroup.

Let X be the space $C_{2\pi}$ of all 2π-periodic continuous functions f, normed by $\|f\|_C = \sup_{-\pi \leqslant x \leqslant \pi} |f(x)|$. We define a nonlinear operator A in $C_{2\pi}$ by

$$D(A) = \text{Domain of } A = \{f; f, f', f'' \in C_{2\pi}\}, \qquad Af = -f'' + F(f'), \tag{6.2}$$

with F defined as above. In [18] it is shown that A is m-accretive, i.e., A is accretive and $R(I + \lambda A) = C_{2\pi}(\lambda > 0)$. Hence $(-A)$ generates a semigroup $T \in Q(C_{2\pi})$ (which cannot be described explicitly) in the sense of Theorem 2.1. Concerning the order of magnitude of $\| T(t)f - f \|$ $(f \in C_{2\pi})$ with respect to t, we may apply the results of part I to T, in particular those expressed by the K-functional

$$K(t,f) = K(t,f;\, C_{2\pi}, D(A)) := \inf_{g \in D(A)} (\| f - g \|_C + t \| Ag \|_C).$$

For a further, more concrete characterization of the K-functional, we will compare it with the second modulus of continuity $\omega_2(t,f)$, which is given by

$$\omega_2(t,f) = \sup_{0 \leqslant s \leqslant t} \| f(\cdot - s) - 2f(\cdot) + f(\cdot + s) \|_C .$$

In the course of proof we also need the first modulus of continuity

$$\omega_1(t,f) = \sup_{0 \leqslant |s| \leqslant t} \| f(\cdot + s) - f(\cdot) \|_C .$$

LEMMA 6.1. *Under the above hypotheses there hold the following inequalities for $f \in C_{2\pi}$ and $0 < t \leqslant 1$, c and M being positive constants*:

$$K(t^2,f) \leqslant \tfrac{3}{2}\omega_2(t,f) + t^2 \sup \left\{ | F(v) |;\, | v | \leqslant \left[\omega_1(1,f) + \int_t^1 \omega_2(s,f)\, s^{-2}\, ds \right] \right\}, \tag{6.3}$$

$$\omega_2(t,f) \leqslant 4K(t^2,f) + t^2 \sup\{ | F(v) |;\, | v | \leqslant 2\pi \| f \| + 2\pi t^{-2} K(t^2,f) \}. \tag{6.4}$$

In particular, if $F(v) = O(| v |^r)$, $r > 0$, then

$$K(t^2,f) \leqslant \tfrac{3}{2}\omega_2(t,f) + t^2 c^r M \left[\omega_1(1,f) + \int_t^1 \omega_2(s,f)\, s^{-2}\, ds \right]^r; \tag{6.5}$$

if $| F(v) | = | v |^r$, $r > 0$, then

$$\omega_2(t,f) \leqslant 4K(t^2,f). \tag{6.6}$$

Proof. If $f \in C_{2\pi}$, $0 < t \leqslant 1$, then

$$K(t^2,f) \leqslant \| f - g_t \|_C + t^2 \| g_t'' \|_C + t^2 \| F(g_t') \|_C ,$$

where

$$g_t(x) := t^{-2} \int_{-t/2}^{t/2} \int_{-t/2}^{t/2} f(x + \tau_1 + \tau_2)\, d\tau_1\, d\tau_2$$

belongs to $D(A)$. Since

$$f(x) - g_t(x) = -(2t^2)^{-1} \int_{-t/2}^{t/2} \int_{-t/2}^{t/2} [f(x+\tau_1+\tau_2) + f(x-\tau_1-\tau_2) - 2f(x)]\, d\tau_1\, d\tau_2,$$

and

$$g_t''(x) = t^{-2}[f(x+t) + f(x-t) - 2f(x)],$$

one has

$$\| f - g_t \|_C \leqslant \tfrac{1}{2}\omega_2(t, f) \tag{6.7}$$

and

$$\| g_t'' \|_C \leqslant t^{-2}\omega_2(t, f). \tag{6.8}$$

Concerning the estimation of $\| F(g_t')\|_C$, note that

$$\| g_t' \|_C \leqslant t^{-1}\omega_1(t, f) \leqslant c \left[\omega_1(1, f) + \int_t^1 \omega_2(s, f)\, s^{-2}\, ds\right], \tag{6.9}$$

the latter inequality being known as Marchaud's inequality (see, e.g., [29]). Since F is continuous on $\mathbb{R}^1$, (6.9) implies

$$\| F(g_t')\|_C \leqslant \sup \left\{| F(v)|;\ | v | \leqslant c \left[\omega_1(1, f) + \int_t^1 \omega_2(s, f)\, s^{-2}\, ds\right]\right\}. \tag{6.10}$$

Thus, combining (6.7), (6.8), and (6.10), inequality (6.3) follows. In the special case that $F(v) = O(| v |^r)$ (6.10) implies

$$\| F(g_t')\|_C \leqslant c^r M \left[\omega_1(1, f) + \int_t^1 \omega_2(s, f)\, s^{-2}\, ds\right]^r,$$

yielding (6.5). Concerning (6.4), one has for each $g \in D(A)$

$$\omega_2(t, f) \leqslant \omega_2(t, f - g) + \omega_2(t, g). \tag{6.11}$$

Now,

$$\omega_2(t, f - g) \leqslant 4 \| f - g \|_C \tag{6.12}$$

and

$$\omega_2(t, g) \leqslant t^2 \| g'' \|_C \leqslant t^2 \| Ag \|_C + t^2 \| F(g')\|_C . \tag{6.13}$$

To estimate $\| F(g')\|_C$, we use the inequality (see [18, Lemma 2]):

$$\| g' \|_C \leqslant 2\pi \| g + Ag \|_C ,$$

which may be estimated from above by

$$2\pi \| f \|_C + 2\pi t^{-2}[\| f - g \|_C + t^2 \| Ag \|_C]$$

for $0 < t \leqslant 1$. Thus one obtains

$$\| F(g') \|_C \leqslant \sup\{| F(v) |;\ | v | \leqslant 2\pi \| f \|_C + 2\pi t^{-2}[\| f - g \|_C + t^2 \| Ag \|_C]\}. \quad (6.14)$$

Combining (6.12) and (6.13) and noting (6.14), (6.11) yields

$$\omega_2(t, f) \leqslant 4[\| f - g \|_C + t^2 \| Ag \|_C] \\ + t^2 \sup\{| F(v) |;\ | v | \leqslant 2\pi \| f \|_C + 2\pi t^{-2}[\| f - g \|_C + t^2 \| Ag \|_C]\}.$$

Taking the infimum with respect to all $g \in D(A)$, (6.4) follows. To obtain (6.5) if $| F(v) | = | v |^r$, (6.14) is replaced by

$$\| F(g') \|_C \leqslant \| Ag \|_C . \quad (6.15)$$

This inequality may be easily checked by the following consideration. Since F and g' are continuous, there exists $x_0 \in [-\pi, \pi]$ such that $| F(g'(x)) |$ has an absolute maximum at x_0, i.e.,

$$\| F(g') \|_C = | F(g'(x_0)) | = | g'(x_0) |^r.$$

Without loss of generality we may assume $| F(g'(x_0)) | > 0$ which is equivalent with $g'(x_0) \neq 0$. Then $| F(g'(x)) |$ has a derivative at x_0 which equals zero, i.e.

$$\left| \frac{d}{dx} | F(g'(x)) | \right|_{x = x_0} = r \, | g'(x_0) |^{r-1} \, | g''(x_0) | = 0.$$

This implies $g''(x_0) = 0$. Thus

$$\| F(g') \|_C = | -g''(x_0) + F(g'(x_0)) | \leqslant \| -g'' + F(g') \|_C ,$$

establishing (6.15). The proof of the lemma is now complete.

If $\omega_2(t, f) = O(t^{2\alpha})$, $0 < \alpha \leqslant 1$, then

$$c \left[\omega_1(1, f) + \int_t^1 \omega_2(s, f)\, s^{-2}\, ds \right] = O(t^{\min(0, 2\alpha - 1)}) \qquad (\alpha \neq \tfrac{1}{2}) \\ = O(t^{-\kappa}) \qquad (\alpha = \tfrac{1}{2}),$$

any κ, $0 < \kappa < \infty$. This implies by (6.3):

$$K(t, f) = O(t^\alpha) + t \sup[| F(v) |;\ | v | = O(t^{\min(0, \alpha - 1/2)})] \qquad (\alpha \neq \tfrac{1}{2})$$

$$K(t, f) = O(t^\alpha) + t \sup[| F(v) |;\ | v | = O(t^{-\kappa})] \qquad (\alpha = \tfrac{1}{2}).$$

Hence, if $\frac{1}{2} < \alpha \leqslant 1$, then $K(t, f) = O(t^\alpha)$. If in addition $F(v) = O(|v|^r)$, $r > 0$, (6.5) implies for all α, $0 < \alpha \leqslant 1$:

$$K(t, f) = O(t^\alpha + t^{1+\min(0,\alpha-1/2)r}) \qquad (\alpha \neq \tfrac{1}{2})$$

$$K(t, f) = O(t^{1/2}) \qquad (\alpha = \tfrac{1}{2})$$

$$K(t, f) = O(t^\alpha) \qquad (0 \leqslant r \leqslant 2,\ 0 < \alpha \leqslant 1 \text{ or } r > 2,$$
$$(r - 2)/[2(r - 1)] \leqslant \alpha \leqslant 1).$$

Conversely, if $K(t, f) = O(t^\alpha)$, $0 < \alpha \leqslant 1$, then (6.4) yields

$$\omega_2(t, f) = O(t^{2\alpha}) + t^2 \sup\{|F(v)|; |v| = O(t^{2\alpha-2})\}.$$

This estimate implies $\omega_2(t, f) = O(t^2)$ if $\alpha = 1$. For the special case in which $|F(v)| = |v|^r$ $(r > 0)$, (6.6) yields $\omega_2(t, f) = O(t^{2\alpha})$ for all α, $0 < \alpha \leqslant 1$. Combining these results with those of Theorem 4.3, we obtain the following theorems which, for simplicity, are only formulated for the case $q = \infty$, the counterpart for $1 \leqslant q < \infty$ then being obvious. Here $\mathrm{Lip}_2\, \beta$ $(0 < \beta \leqslant 2)$ denotes the class of functions $f \in C_{2\pi}$ for which $\sup_{0<t\leqslant 1} t^{-\beta}\omega_2(t, f)$ is finite.

THEOREM 6.2. *Let A be defined by* (6.2) *and let $T = \{T(t); t \geqslant 0\}$ be the semigroup generated by $(-A)$.*

(a) *The following assertions are equivalent for an $f \in C_{2\pi}$:*

(i) $\| T(t)f - f \|_C = O(t)$; (ii) $f \in \mathrm{Lip}_2\, 2$; (iii) *f, f' are absolutely continuous and periodic and $f'' \in L^\infty_{2\pi}$.*

(b) *If $f \in \mathrm{Lip}_2\, 2\alpha$ $(\frac{1}{2} < \alpha < 1)$, then $\| T(t)f - f \|_C = O(t^\alpha)$.*

Part (a) of the theorem describes the optimal approximation behavior of T. For the equivalence of (ii) and (iii) see [6, p. 129]. Part (b) is a so-called direct theorem for the nonoptimal case, at least if $\frac{1}{2} < \alpha < 1$. These results hold for an arbitrary continuous function $F(v)$ with $F(0) = 0$. Moreover, if $F(v)$ is specialized as a power function, then even the converse holds in case of nonoptimal approximation.

THEOREM 6.3. *Let $T = \{T(t); t \geqslant 0\}$ be the semigroup generated by $(-A)$ defined in* (6.2) *with $|F(v)| = |v|^r$, $r > 0$.*

(a) *If $0 \leqslant r \leqslant 2$, then for $f \in C_{2\pi}$*

$$\| T(t)f - f \|_C = O(t^\alpha) \qquad (0 < \alpha \leqslant 1) \Leftrightarrow f \in \mathrm{Lip}_2(2\alpha).$$

(b) *If* $r > 2$, *then for* $f \in C_{2\pi}$

$$\| T(t)f - f \|_C = O(t^\alpha) \qquad \left(\frac{r-2}{2(r-1)} \leqslant \alpha \leqslant 1\right) \Leftrightarrow f \in \mathrm{Lip}_2(2\alpha).$$

For $0 \leqslant r \leqslant 2$ this theorem is an equivalence theorem for all values of α in question, while for $r > 2$ the region of admissible α depends on r, however, always including the range $\frac{1}{2} \leqslant \alpha \leqslant 1$.

It is well known (see [6, p. 127]) that the classes $\mathrm{Lip}_2(2\alpha)$ $(0 < \alpha \leqslant 1)$, which are linear, completely characterize the approximation behavior of the linear semigroup $W = \{W(t); t \geqslant 0\}$ which is associated with the periodic singular integral of Weierstrass

$$[W(t)f](x) = \frac{1}{2\pi} \int_0^{2\pi} f(u)\, \theta_3(x-u; t)\, du \qquad (t > 0, f \in C_{2\pi}),$$

$\theta_3(x, t) = \sum_{k=-\infty}^{\infty} e^{-k^2 t} e^{ikx}$ being Jacobi's theta-function. Thus, in the cases described by Theorem 6.2(a) and 6.3 the approximation behavior of T coincides with that of the semigroup W. In this context see also [25], [30], and [26].

7. Concluding Remarks

As a further application one might discuss the semigroup related to the Cauchy problem

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \Delta \varphi(u(x,t)) \qquad (x \in \Omega \subset \mathbb{R}^n, t > 0) \\ u(x,0) &= f(x) \end{aligned} \tag{7.1}$$

which is treated in the literature under various hypotheses upon the function $\varphi: \mathbb{R}^1 \to \mathbb{R}^1$ and in various function spaces X (see, e.g., [33], [28], [23], [24]). Crandall [11] considered the semigroup associated with (7.1) in the space $L^1(\Omega)$ in case φ is a continuous strictly monotone increasing function on $\mathbb{R}^1$ with $\varphi(0) = 0$ and Ω a bounded region of $\mathbb{R}^n$. Konishi [31] studied the one-dimensional problem on the circle in case $\varphi(v) = v^m$ $(m > 1)$, representing a mathematical model for flow through a homogeneous porous medium. The problem now is the approximation theoretical behavior of the nonlinear semigroup associated with (7.1). The general theory of part I applies, but the details will not be carried out here.

Finally, let us recall that in linear semigroup theory the approximation behavior as treated in this paper is usually described in a topological framework. This is due to the fact that the approximation classes introduced in Definition 4.1 may be supplied with norms by the aid of the functional values $\Phi_{\alpha,q}(\cdot)$ defining them. Hence inequalities between these norms, e.g., (4.2) and (4.3), may be expressed simply as embeddings of the corresponding spaces. Now in the nonlinear situation this interpretation in the setting of normed spaces is of course no longer possible, though the inequalities per se remain valid.

One might try to equip the approximation classes in question with a (nonlinear) metric topology, e.g., the set $[A]^T_{\alpha,q}$ with the metric

$$\rho_{\alpha,q}(f, g) := \| f - g \| + \Phi_{\alpha,q}(\| T(t)f - f - T(t)g + g \|) \qquad (f, g \in [A]^T_{\alpha,q}).$$

But to obtain the topological embeddings wanted for our spaces one would have to restrict the notion of accretiveness by a condition additional to (2.5). This condition would have to be checked in the examples, which may be quite a problem.

References

23. D. G. Aronson, Regularity properties of flows through porous media: the interface, *Arch. Rational Mech. Anal.* **37** (1970), 1–10.
24. D. G. Aronson, Regularity properties of flows through porous media: a counterexample, *SIAM J. Appl. Math.* **19** (1970), 299–307.
25. P. L. Butzer and W. Köhnen, Approximation invariance of semi-group operators under perturbations, *J. Approximation Theory* **2** (1969), 389–393.
26. P. L. Butzer, R. J. Nessel, and W. Trebels, Multipliers with respect to spectral measures in Banach spaces and approximation, I. Radial multipliers in connection with Riesz-bounded spectral measures, *J. Approximation Theory* **8** (1973), 335–356.
27. G. E. Forsythe and W. R. Wasow, "Finite-Difference Methods for Partial Differential Equations," John Wiley and Sons, New York, 1960.
28. S. I. Hudjaev and A. I. Vol'pert, Cauchy's problem for degenerate second order parabolic equations, *Math. USSR-Sb.* **7** (1969), 365–387.
29. H. Johnen, Inequalities connected with the moduli of smoothness, *Mat. Vesnik* **9**, No. 24 (1972), 289–303.
30. W. Köhnen, Das Anfangswertverhalten von Evolutionsgleichungen in Banachräumen, Teil I: Approximationseigenschaften von Evolutionsoperatoren; Teil II: Anwendungen, *Tôhoku Math. J.* **22** (1970), 566–596; **23** (1971), 621–639.
31. Y. Konishi, A remark on fluid flows through porous media, *Proc. Japan Acad.* **49** (1973), 20–24.
32. S. N. Kruzhkov, Generalized solutions of nonlinear equations of first order with several variables, *Mat. Sb.* **70** (1966), 394–415.
33. O. A. Oleinik, On some degenerate quasilinear parabolic equations, Seminari dell Istituto Nazionale di Alta Math. 1962–63, Oderisi, 1964, pp. 355–371.